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Wess-Zumino model in the causal approach

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Abstract. The Wess–Zumino model is analysed in the framework of the causal approach of Epstein–Glaser. The condition of invariance with respect to supersymmetry transformations is similar to gauge invariance in the Zürich formulation. We prove that this invariance condition can be implemented in all orders of perturbation theory, i.e. the anomalies are absent in all orders. This result is of a purely algebraic nature. We work consistently in the quantum framework based on the Bogoliubov axioms of perturbation theory, so no Grassmann variables are necessary.

1 Introduction

The causal approach to renormalization theory of Epstein and Glaser [8,9] seems to be the most convenient way to understand renormalization theory at the fundamental level. It is also extremely useful for purely computational reasons. In this paper we will prove that supersymmetric theories can be also studied in a completely rigorous way in this framework. We will analyse the simplest supersymmetric model, namely the Wess–Zumino model [22,15]. We do not use in this paper the superfield formulation [18,21, 20]. We prefer to formulate this model working directly in the quantum framework: we consider in the Fock space of the model (generated by a scalar, a pseudo-scalar and a Majorana quantum free field of the same positive mass) and construct the chronological products verifying the Bogoliubov axioms. We can define in this Fock space the supersymmetric current and the supercharge; they are only the linear contributions of the usual expressions appearing in the literature. Then we impose the condition of supersymmetry invariance at the quantum level in close analogy to the condition of gauge invariance adopted by the Zürich group for gauge theories [6,7]; the physical meaning of this condition is the invariance of the S-matrix with respect to supersymmetric transformations in the adiabatic limit. In the next section we give the essential points concerning the perturbation theory in the sense of Bogoliubov and Epstein–Glaser; for more details see [12] and the literature cited there. In Sect. 3 we define the Wess–Zumino model in this framework and in Sect. 4 we prove that supersymmetry invariance can be implemented to all orders of perturbation theory by the purely algebraic procedure of distribution splitting. We mention that this problem has been studied in the framwork of the quantum Noether method (closely related to the causal approach) in [14] for the case of zero mass; our approach has the virtue of being quite elementary and using only the basic facts concerning the Epstein–Glaser approach to renormalization theory. We add a few comments about the conservation of the supersymmetric current at the end of this paper. The absence of anomalies in all orders has been proved using the usual BRST approach in [13,3,2].

2 Perturbation theory in the causal approach

2.1 Bogoliubov axioms

Let us recall briefly the main ideas of the Epstein–Glaser–Scharf approach. According to Bogoliubov and Shirkov, the S-matrix is constructed inductively order by order as a formal series of operator-valued distributions:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{\mathrm{i}^n}{n!} \int_{\mathbb{R}^{4n}} \mathrm{d}x_1 \cdots \mathrm{d}x_n \times T(x_1, \cdots, x_n) g(x_1) \cdots g(x_n), \qquad (2.1.1)$$

where $g(x)$ is a tempered test function in the Minkowski space $\mathbb{R}^4$ that switches the interaction and $T(X) \equiv T(x_1, \cdots, x_n)$ are operator-valued distributions acting in the Hilbert space $\mathcal{H}$ generated by some collection of free fields. These operator-valued distributions, which are called *chronological products* should verify some properties which can be argued starting from *Bogoliubov axioms*. We give here the set of axioms imposed on the chronological products following [9].

(1) Domain: There is a common dense domain of definition $D_0 \in \mathcal{F}$ for all chronological products.
(2) Symmetry:

$$T(x_{\pi(1)}, \cdots x_{\pi(n)}) = T(x_1, \cdots x_n), \quad \forall \pi \in \mathcal{P}_n. \qquad (2.1.2)$$

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(3) Poincaré invariance: There exists in the Fock space of the model an unitary representation $(a, A) \mapsto U_{a,A}$ of the group $\in SL(2,\mathbb{C})$ (the universal covering group of the proper orthochronous Poincaré group $\mathcal{P}^{\uparrow}_{+}$; see [19] for notation) such that

$$U_{a,A} T(x_1, \cdots, x_n) U_{a,A}^{-1} = T(\delta(A)\cdot x_1 + a, \cdots, \delta(A)\cdot x_n + a), \quad \forall A \in SL(2,\mathbb{C}), \forall a \in \mathbb{R}^4, \tag{2.1.3}$$

where $SL(2,\mathbb{C}) \ni A \mapsto \delta(A) \in \mathcal{P}^{\uparrow}_{+}$ is the covering map. In particular, *translation invariance* is essential for implementing the Epstein–Glaser scheme of renormalization.
Sometimes it is possible to supplement this axiom by invariance properties with respect to space-time inversions, charge conjugation or invariance with respect to some global group of transformations (continuous or discrete). In this paper we will impose the invariance with respect to supersymmetry transformations.

(4) Causality: We use the standard notations: $V^{\pm} \equiv \{x \in \mathbb{R}^4 |\ x^2 > 0,\ \mathrm{sign}(x_0) = \pm\}$ for the upper (lower) light-cone and $\overline{V^{\pm}}$ for their closures. If $X \equiv \{x_1, \cdots, x_m\} \in \mathbb{R}^{4m}$ and $Y \equiv \{y_1, \cdots, y_n\} \in \mathbb{R}^{4m}$ are such that $x_i - y_j \notin \overline{V^-},\ \forall i = 1,\dots,m,\ j = 1,\dots,n$, we use the notation $X \geq Y$. If $x_i - y_j \notin \overline{V^+} \cup \overline{V^-},\ \forall i = 1,\dots,m,\ j = 1,\dots,n$, we use the notations $X \sim Y$. Then the causality axiom can be written as follows:

$$T(X_1 X_2) = T(X_1) T(X_2), \quad \forall X_1 \geq X_2. \tag{2.1.4}$$

(5) Unitarity: We define the expressions

$$(-1)^{|X|}\bar{T}(X) \equiv \sum_{r=1}^{|X|} (-1)^r \sum_{X_1,\cdots,X_r \in \mathrm{Part}(X)} T(X_1)\cdots T(X_r). \tag{2.1.5}$$

One calls the operator-valued distributions $T(\bar{X})$ *anti-chronological products*. Then the unitarity axiom is

$$\bar{T}(X) = T(X)^*, \quad \forall X. \tag{2.1.6}$$

2.2 Epstein–Glaser induction

In this subsection we summarize the steps of the inductive construction of Epstein and Glaser [8]. The main point is a careful formulation of the induction hypothesis. So, we suppose that we have the *interaction Lagrangian* $T(x)$ given by a sum of Wick monomials acting in a certain Fock space. We make the simplifying assumption (valid for the Wess–Zumino model) that *no derivative of the fields appear* in the Wick monomials composing $T(x)$. Moreover, we require the following properties:

$$U_{a,A} T(x) U_{a,A}^{-1} = T(\delta(A)\cdot x + a), \quad \forall A \in SL(2,\mathbb{C}), \tag{2.2.1}$$

$$[T(x), T(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad s.t. \quad x \sim y, \tag{2.2.2}$$

and

$$T(x)^* = T(x). \tag{2.2.3}$$

Usually, these requirements are supplemented by covariance with respect to some discrete symmetries like space-time inversions, charge conjugations or global invariance with respect to some Lie group of symmetry. In this paper we will add supersymmetry invariance (see the next section).

Let us define the *degree* of a Wick monomial $\deg(W)$ by assigning to every integer spin field factor and every derivative the value 1, for every half-integer spin field factor the value 3/2 and summing over all factors. We consider the interaction Lagrangian to have the canonical dimension ≤ 4.

We suppose that we have constructed the chronological products $T(X),\ |X| \leq n-1$ having the properties (2.1.2)–(2.1.4) and (2.1.6). We add to the induction hypothesis the following *Wick expansion* property:

$$T(X) = \sum_i t_i(X) W_i(X), \quad |X| \leq n-1, \tag{2.2.4}$$

where $W_i(X)$ are the basis of linearly independent Wick monomials *without derivatives on the fields* and $t_i(X)$ are numerical distributions; they are called *renormalized Feynman amplitudes* and are Poincaré covariant. Finally, the following limitation is included in the induction hypothesis:

$$\omega(t_i) + \deg(W_i) \leq 4, \quad \forall i, \tag{2.2.5}$$

where by $\omega(t)$ we mean the order of the singularity of the distribution t (see [16] for the definition).

Let us note that in this case we also have

$$[T(X_1), T(X_2)] = 0, \quad \text{if} \quad X_1 \sim X_2, \quad |X_1| + |X_2| \leq n-1. \tag{2.2.6}$$

We want to construct the distribution-valued operators $T(X),\ |X| = n$, such that the induction hypothesis stays true.

Here are the main steps of the induction proof.

(1) One constructs from $T(X),\ |X| \leq n-1$ the expressions $\bar{T}(X),\ |X| \leq n-1$ according to (2.1.5).
(2) One defines for $|X| = n$ the expressions:

$$A'(X) \equiv \sum_{\substack{X_1, X_2 \in \mathrm{Part}(X) \\ X_2 \neq \emptyset, x_n \in X_1}} (-1)^{|X_2|} T(X_1)\bar{T}(X_2), \tag{2.2.7}$$

$$R'(X) \equiv \sum_{\substack{X_1, X_2 \in \mathrm{Part}(X) \\ X_2 \neq \emptyset, x_n \in X_1}} (-1)^{|X_2|} \bar{T}(X_2) T(X_1), \tag{2.2.8}$$

and

$$D(X) \equiv A'(X) - R'(X). \tag{2.2.9}$$

Then one can prove that we have the causal support property:

$$\mathrm{supp}(D(X)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n), \tag{2.2.10}$$

where we use standard notation:

$$\Gamma^{\pm}(x_n) \equiv \{(x_1,\dots,x_n) \in (\mathbb{R}^4)^n | x_i - x_n \in V^{\pm}, \forall i = 1,\dots,n-1\}. \tag{2.2.11}$$

(3) The distribution $D(X)$ can be written as a sum

$$D(X) = \sum_i d_i(X) W_i(X), \tag{2.2.12}$$

where $d_i(X)$ are numerical distributions with causal support, i.e.,

$$\mathrm{supp}(d_i(X)) \subset \Gamma^+(x_n) \cup \Gamma^-(x_n), \tag{2.2.13}$$

and they are Poincaré covariant. Finally, the following limitations are valid:

$$\omega(d_i) + \deg(W_i) \leq 4, \quad \forall i. \tag{2.2.14}$$

Let us note that in theories with derivatives it is much more difficult to extract the properties of the numerical distributions d_i from the corresponding properties of the operatorial distribution $D(X)$: one has a supplementary induction hypothesis concerning the Wick submonomials [8,5].

(4) Now we have the following result from [7,16]: Let d be a $SL(2,\mathbb{C})$-covariant distribution with causal support. Then, there exists a causal splitting

$$d = a - r, \quad \mathrm{supp}(a) \subset \Gamma^+(x_n), \quad \mathrm{supp}(r) \subset \Gamma^-(x_n), \tag{2.2.15}$$

which is also $SL(2,\mathbb{C})$-covariant and such that

$$\omega(a) \leq \omega(d), \quad \omega(r) \leq \omega(d). \tag{2.2.16}$$

So, there exists a $SL(2,\mathbb{C})$-covariant causal splitting:

$$D(X) = A(X) - R(X), \quad |X| = n, \tag{2.2.17}$$

with $\mathrm{supp}(A(X)) \subset \Gamma^+(x_n)$ and $\mathrm{supp}(R(X)) \subset \Gamma^-(x_n)$.

For that reason, the expressions $A(X)$ and $R(X)$ are called *advanced* (respectively *retarded*) products.

(5) One can prove that the following relation is true:

$$D(X)^* = (-1)^{n-1} D(X), \quad |X| = n. \tag{2.2.18}$$

As a consequence, the causal splitting obtained above can be chosen such that

$$A(X)^* = (-1)^{n-1} A(X). \tag{2.2.19}$$

This can be done by the redefinition

$$A(X) \to \frac{1}{2}\left[A(X) + (-1)^{n-1} A(X)^*\right], \tag{2.2.20}$$

which does not affect the support property.

(6) Let us define

$$T'(X) \equiv A(X) - A'(X) = R(X) - R'(X). \tag{2.2.21}$$

Then these expressions satisfy the $SL(2,\mathbb{C})$-covariance, causality and unitarity conditions (2.1.3), (2.1.4), (2.1.6) and the Wick expansion property. If we substitute

$$T(x_1, \cdots, x_n) \to \frac{1}{n!} \sum_\pi T'(x_{\pi(1)}, \cdots, x_{\pi(n)}), \tag{2.2.22}$$

where the sum runs over all permutations of the numbers $\{1, \ldots, n\}$, then we also have the symmetry axiom (2.1.2).

The solution of the renormalization problem is not unique. If all chronological products up to order $n-1$ are given, then the non-uniqueness in order n is given by the possibility of adding to the distributions $T(X),\ |X| = n$ some finite renormalizations (quasi-local operators in the terminology of [1]) $N(X)$.

We mention in closing this section that one can construct more general chronological producs [17,5], i.e., if $A_i(x),\ i = 1, \ldots, n$, are some Wick polynomials, then one can give a natural system of axioms for the chronological products $T(A_1(x_1), \ldots, A_n(x_n))$. It is obvious how to generalise the previous axioms to this case. We only mention that the symmetry axiom must take into account the existence of Fermi fields: if the Wick monomial $A_i(x)$ has f_i Fermi fields, then the commutation of $A_i(x_i)$ with $A_j(x_j)$ in the expression $T(A_1(x_1), \ldots, A_n(x_n))$ produces a sign $(-1)^{f_i f_j}$.

The connection with the chronological product defined above is given by $T(x_1, \cdots, x_n) \equiv T(T(x_1), \ldots, T(x_n))$.

3 Wess–Zumino model

3.1 The definition of the model

In this subsection we define the Wess–Zumino model in the framework of the Bogoliubov axioms presented above. We consider the Hilbert space $\mathcal{H}$ endowed with the scalar product $\langle\cdot,\cdot\rangle$ and generated by applying to the vacuum Ω the following free fields: the scalar field $A(x)$, the pseudoscalar field $B(x)$ and the Majorana field $\psi(x)$. These fields are assumed to have the same mass $m > 0$.

To describe the Majorana field we need Dirac matrices $\gamma^\mu,\ \mu = 0, \ldots, 3$ for which we prefer the chiral representation:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1,2,3; \tag{3.1.1}$$

here $\sigma_i,\ i = 1,2,3$ are the Pauli matrices. This is a representations in which the matrix $\gamma_5 \equiv \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ is diagonal:

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1.2}$$

If $u \in \mathbb{C}^4$ is a spinor considered as a column vector then we define $\bar{u} \equiv u^* \gamma_0$ considered as a row vector.

The fields considered in our model are determined by the following properties:

(1) Equations of motion:

$$(\partial^2 + m^2)A(x) = 0, \quad (\partial^2 + m^2)B(x) = 0,$$
$$(\mathrm{i}\gamma\cdot\partial - m)\psi(x) = 0. \tag{3.1.3}$$

(2) Canonical (anti-) commutation relations:

$$[A(x), A(y)] = D_m(x-y) \times \mathbf{1},$$
$$[B(x), B(y)] = D_m(x-y) \times \mathbf{1},$$
$$\{\psi_\alpha(x), \psi_\beta(y)\} = (S_m(x-y)C)_{\alpha\beta} \times \mathbf{1}. \tag{3.1.4}$$

and all other (anti-) commutators are null; here $C = \gamma_0\gamma_2$ is the charge conjugation matrix and $S_m(x),\ m \geq 0$ is a 4×4 matrix given by

$$S_m(x) \equiv (\mathrm{i}\gamma \cdot \partial + m) D_m(x). \tag{3.1.5}$$

(3) $SL(2,\mathbb{C})$-covariance:

$$\begin{aligned} U_{a,A} A(x) U_{a,A}^{-1} &= A(\delta(A)\cdot x + a),\\ U_{a,A} B(x) U_{a,A}^{-1} &= B(\delta(A)\cdot x + a),\\ U_{a,A} \psi(x) U_{a,A}^{-1} &= S(A^{-1})\psi(\delta(A)\cdot x + a), \end{aligned} \tag{3.1.6}$$

here $\delta : SL(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ is the covering map and

$$S(A) \equiv \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^* \end{pmatrix}. \tag{3.1.7}$$

(4) Space-time covariance:

$$\begin{aligned} U_{I_s} A(x) U_{I_s}^{-1} &= A(I_s\cdot x),\\ U_{I_s} B(x) U_{I_s}^{-1} &= -B(I_s\cdot x),\\ U_{I_s} \psi(x) U_{I_s}^{-1} &= \mathrm{i}\gamma_0 \psi(I_s\cdot x). \end{aligned} \tag{3.1.8}$$

$$\begin{aligned} U_{I_t} A(x) U_{I_t}^{-1} &= A(I_t\cdot x),\\ U_{I_t} B(x) U_{I_t}^{-1} &= B(I_t\cdot x),\\ U_{I_t} \psi(x) U_{I_t}^{-1} &= C^{-1}\gamma_5 \psi(I_t\cdot x). \end{aligned} \tag{3.1.9}$$

The space-time inversion is $U_{I_{st}} \equiv U_{I_s} U_{I_t}$.

(5) Hermitian conjugation properties:

$$A_{a\mu}(x)^* = A_{a\mu}(x), \quad B(x)^* = B(x), \quad \psi(x)^c = \psi(x), \tag{3.1.10}$$

where $*$ is the conjugation with respect to the scalar product $\langle\cdot,\cdot\rangle$ and the definition of the charge conjugate of the spinor $u \in \mathbb{C}^4$ is

$$u^c \equiv C\bar{u}^T. \tag{3.1.11}$$

(6) Charge conjugation invariance:

$$\begin{aligned} &U_C A(x) U_C^{-1} = A(x), \quad U_C B(x) U_C^{-1} = B(x),\\ &U_C \psi(x) U_C^{-1} = C\bar{\psi(x)}^T = \psi(x). \end{aligned} \tag{3.1.12}$$

(7) Moreover, we suppose that these operators are leaving the vacuum invariant:

$$U_{a,A}\Omega = \Omega, \quad U_{I_s}\Omega = \Omega, \quad U_{I_t}\Omega = \Omega, \quad U_C\Omega = \Omega. \tag{3.1.13}$$

Let us make a remark. One can prove that the operators $U_{a,A},\ U_{I_s}$ and U_{I_t} are realizing a projective representation of the Poincaré group; i.e., they have suitable commutation properties. Also the charge conjugation operator commutes with these operators. As is known, there is some freedom in choosing some phases in the definitions of the spatial and temporal inversions; we have made the convenient choice which ensures this commutativity property.

In this Fock space we can define the spinorial operator $J_\alpha^\mu(x) \equiv J_\alpha^\mu(x)_{\alpha=1}^4$ called the *supercurrent* according to the formula

$$\begin{aligned} J^\mu \equiv\ &:\partial_\nu A \gamma^\nu\gamma^\mu\psi: + \mathrm{i} :\partial_\nu B\gamma_5\gamma^\nu\gamma^\mu\psi: + \mathrm{i}m : A\gamma^\mu\psi :\\ &+ \mathrm{i} : B\gamma_5\gamma^\mu\psi : \end{aligned} \tag{3.1.14}$$

where the interpretation of this operator as a column vector with four components is obvious. Then we have by direct computation, using the equations of motion, the following *conservation law*:

$$\partial_\mu J^\mu = 0. \tag{3.1.15}$$

Moreover, the supercurrent, considered as a spinor, is charge conjugation invariant:

$$(J^\mu)^c = J^\mu. \tag{3.1.16}$$

One can define formally the *supercharges* as a four-component operator according to

$$Q_\alpha = \int_{\mathbb{R}^3} \mathrm{d}^3x J_\alpha^0(x). \tag{3.1.17}$$

To avoid problems connected with the existence of the integral, it is better to work in momentum space. One has the standard expressions of the free fields considered in the model:

$$A(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_m^+} \mathrm{d}\alpha_m^+(p) \left[\mathrm{e}^{-\mathrm{i}p\cdot x} a(p) + \mathrm{e}^{\mathrm{i}p\cdot x} a^*(p)\right], \tag{3.1.18}$$

$$B(x) \equiv \frac{1}{(2\pi)^{3/2}} \int_{X_m^+} \mathrm{d}\alpha_m^+(p) \left[\mathrm{e}^{-\mathrm{i}p\cdot x} b(p) + \mathrm{e}^{\mathrm{i}p\cdot x} b^*(p)\right], \tag{3.1.19}$$

and

$$\begin{aligned} \psi(x) \equiv &\frac{1}{(2\pi)^{3/2}} \int_{X_m^+} \mathrm{d}\alpha_m^+(p)\\ &\times \sum_{s=1}^2 \left[\mathrm{e}^{-\mathrm{i}p\cdot x} u_s(p) d_s(p) + \mathrm{e}^{\mathrm{i}p\cdot x} u_s^c(p) d_s^*(p)\right], \end{aligned} \tag{3.1.20}$$

where $u_s(p)$ are two independent solutions of positive energy of the free Dirac equation properly normalized. Here X_m^+ is the upper hyperboloid of mass m and $\alpha_m^+(p)$ is the Lorentz invariant measure defined on this Borel set. One can see that the formal integration of the formula (3.1.17) gives

$$\begin{aligned} Q \equiv &\sum_{s=1}^2 \int_{X_m^+} \mathrm{d}\alpha_m^+(p)\\ &\times\{-\mathrm{i}[a(p) u_s^c(p) d_s^*(p) - a^*(p) u_s(p) d(p)]\\ &+\gamma_5[b(p) u_s^c(p) d_s^*(p) - b^*(p) u_s(p) d_s(p)]\}, \end{aligned} \tag{3.1.21}$$

which is a perfectly well-defined expression acting in the Fock space and it will be taken as a definition. It is elementary to obtain the following (anti-) commutation relations:

$$[Q, a(p)] = -\mathrm{i} \sum_s u_s(p) d_s(p),$$

$$
\begin{aligned}
[Q, a^*(p)] &= -\mathrm{i} \sum_s u_s^c(p) d_s^*(p), \\
[Q, b(p)] &= \gamma_5 \sum_s u_s(p) d_s(p), \\
[Q, a^*(p)] &= \gamma_5 \sum_s u_s^c(p) d_s^*(p), \\
\{Q, d_s^*(p)\} &= \mathrm{i}[a^*(p) + \mathrm{i}\gamma_5 b^*(p)] u_s(p), \\
\{Q, d_s(p)\} &= -\mathrm{i}[a(p) + \mathrm{i}\gamma_5 b(p)] u_s(p). \qquad (3.1.22)
\end{aligned}
$$

and

$$Q^c = Q, \qquad (3.1.23)$$

$$U_{a,A} Q = Q U_{a,A}. \qquad (3.1.24)$$

We will need the relations (3.1.22) in the coordinate space:

$$
\begin{aligned}
&[Q, A(x)] = -\mathrm{i}\psi(x), \quad [Q, B(x)] = \gamma_5 \psi(x), \\
&\{Q_\alpha, \psi_\beta(x)\} = -\partial_\mu A(x)(\gamma^\mu C)_{\alpha\beta} - \mathrm{i}\partial_\mu B(x)(\gamma_5\gamma^\mu C)_{\alpha\beta} \\
&\qquad - \mathrm{i} m A(x) C_{\alpha\beta} + m B(x)(\gamma_5 C)_{\alpha\beta}. \qquad (3.1.25)
\end{aligned}
$$

We also can prove that the supersymmetry algebra is verified: first we have

$$U_{a,A} Q_\alpha U_{a,A}^{-1} = S(A)_{\alpha\beta} Q_\beta, \qquad (3.1.26)$$

where $A \to S(A)$ is the representation $(0,1/2) \oplus (1/2,0)$ of the group $SL(2,\mathbb{C})$ corresponding to the Majorana representation of the Dirac matrices. Next

$$\{Q_\alpha,\ Q_\beta\} = -2\gamma^\mu_{\alpha\beta} P_\mu, \qquad (3.1.27)$$

where P_μ are the translation generators.

As we have said in the preceding section, the Bogoliubov construction of the perturbation series starts with the first order term $T(x)$. We have the following result:

Proposition 3.1. Let us define the operator

$$
\begin{aligned}
T(x) \equiv c_1 \Big[&m : A(x)^3 : + m : A(x) B(x)^2 : \\
&+ : \bar\psi(x)\psi(x) A(x) : - \mathrm{i} : \bar\psi(x)\gamma_5\psi(x) B(x) : \Big] \\
&+ c_2 \Big[m^2 : A(x)^2 : + m^2 : B(x)^2 : \\
&+ \frac{1}{2} m : \bar\psi(x)\psi(x) : \Big] \qquad (3.1.28)
\end{aligned}
$$

and the spinor operator

$$
\begin{aligned}
T^\mu(x) \equiv c_1 &\left[-\mathrm{i} : A(x)^2 \gamma^\mu \psi(x) : + \mathrm{i} : B(x)^2 \gamma^\mu \psi(x) : \right. \\
&\left. + 2 : A(x) B(x) \gamma_5 \gamma^\mu \psi(x) : \right] \qquad (3.1.29) \\
&+ c_2 \left[-\mathrm{i} m : A(x) \gamma^\mu \psi(x) : + : B(x) \gamma^\mu \psi(x) : \right]
\end{aligned}
$$

Then the following relation is true:

$$[Q_\alpha, T(x)] = \mathrm{i} \frac{\partial}{\partial x^\mu} T^\mu_\alpha(x). \qquad (3.1.30)$$

Moreover, the most general Wick polynomial of canonical dimension ≤ 4 verifying (3.1.30) is of the type (3.1.28).

As in the case of gauge theories, the relation (3.1.30) expresses the invariance with respect to supersymmetric transformations of the interaction Lagrangian in the formal adiabatic limit. In this particular case, the weak adiabatic limit probably exists due to the fact that the masses of the model are strictly positive. Moreover, the following relations are verified:

(1) $SL(2,\mathbb{C})$-covariance: for any $A \in SL(2,\mathbb{C})$ we have

$$
\begin{aligned}
U_{a,A} T(x) U_{a,A}^{-1} &= T(\delta(A) \cdot x + a), \qquad (3.1.31) \\
U_{a,A} T^\mu(x) U_{a,A}^{-1} &= \delta(A^{-1})^\mu{}_\rho T^\rho(\delta(A) \cdot x + a).
\end{aligned}
$$

(2) Causality:

$$
\begin{aligned}
&[T(x), T(y)] = 0, \quad [T^\mu(x), T^\rho(y)] = 0, \qquad (3.1.32) \\
&[T^\mu(x), T(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad s.t. \quad x \sim y.
\end{aligned}
$$

(3) Unitarity: suppose that $c_1, c_2 \in \mathbb{R}$; then

$$T(x)^* = T(x), \quad T^\mu(x)^c = T^\mu(x). \qquad (3.1.33)$$

Let us notice that there are no derivatives in the expression of the interaction Lagrangian (3.1.28), so we can apply the procedure outlined in the preceding section.

3.2 Second order chronological product

We consider a perturbation theory in the sense of Bogoliubov taking as the interaction Lagrangian the expression (3.1.28) with $c_1 = 1,\ c_2 = 0$.

First, we define some distributions with causal support which will be needed in the next proposition:

$$D_{m,k}(x) \equiv [D_m^{(+)}(x)]^k + (-1)^{k-1} [D_m^{(-)}(x)]^k, \quad \forall k \in \mathbb{N}^*. \qquad (3.2.1)$$

Next, we consider a canonical causal splitting

$$D_{m,k}(x) = D_{m,k}^{\mathrm{adv}}(x) - D_{m,k}^{\mathrm{ret}}(x), \quad \forall k \in \mathbb{N}^*,$$

verifying Lorentz covariance and preserving the order of singularity. By definition, this canonical causal splitting is obtained using the central decomposition formula of [16]. This is possible because all masses are positive. The causal decomposition of $D_{m,1}(x) = D_m(x)$ induces a similar splitting for the distribution

$$S_m(x) = S_m^{\mathrm{adv}}(x) - S_m^{\mathrm{ret}}(x).$$

We will denote the corresponding retarded, advanced and Feynman distributions by $D_{m,k}^{\mathrm{F}}(x)$ and $S_m^{\mathrm{F}}(x)$ respectively. Then we have the following.

Proposition 3.2. The generic form of the second order chronological product is

$$T(x,y) = T^c(x,y) + \delta(x-y) N(x), \qquad (3.2.2)$$

where

$$\begin{aligned}T_2^c(x,y) \equiv&: T(x)T(y) : +6m^2 D^{\rm F}_{m,3}(x-y)\mathbf{1}\\ &-4[(\partial^2-m^2)D^{\rm F}_{m,2}(x-y)] : A(x)A(y) :\\ &-4[(\partial^2-m^2)D^{\rm F}_{m,2}(x-y)] : B(x)B(y) :\\ &+4{\rm i} : \overline{\psi}(x)[\gamma\cdot\partial D^{\rm F}_m(x-y)]\psi(y) :\\ &+9m^2 D^{\rm F}_m(x-y) : A(x)^2A(y)^2 :\\ &+m^2 D^{\rm F}_m(x-y) : B(x)^2B(y)^2 :\\ &+4m^2 D^{\rm F}_m(x-y) : A(x)B(x)A(y)B(y) :\\ &+4 : \overline{\psi}(x)S^{\rm F}_m(x-y)\psi(y)A(x)A(y) :\\ &-4 : \overline{\psi}(x)\gamma_5 S^{\rm F}_m(x-y)\gamma_5\psi(y)B(x)B(y) :\\ &-3m^2 D^{\rm F}_m(x-y)[: A(x)^2B(y)^2 : +(x\leftrightarrow y)]\\ &+3m D^{\rm F}_m(x-y)[: A(x)^2\overline{\psi}(y)\psi(y) : +(x\leftrightarrow y)]\\ &+m D^{\rm F}_m(x-y)[: B(x)^2\overline{\psi}(y)\psi(y) : +(x\leftrightarrow y)]\\ &-2{\rm i}m D^{\rm F}_m(x-y)[: A(x)B(x)\overline{\psi}(y)\gamma_5\psi(y) :\\ &+(x\leftrightarrow y)]\\ &-4{\rm i}[: \overline{\psi}(x)S^{\rm F}_m(x-y)\gamma_5\psi(y)A(x)B(y) :\\ &+(x\leftrightarrow y)]\end{aligned} \tag{3.2.3}$$

and $N(x)$ is a finite normalization.

The proof consists in the explicit computation of the commutator D_2 like in [10]. The contribution $T^c(x,y)$ correspond to the canonical causal splitting of the numerical distributions. It was noticed from the very beginning [22, 15] that the various distributions appearing in the preceding formula have interesting properties: for instance the distribution appearing as the coefficients of $: A(x)A(y) :$, $: B(x)B(y) :$, $: \bar{\psi}(x)\psi(y) :$ and $\mathbf{1}$ are obtained from $D^{\rm F}_{m,2}$ and $D^{\rm F}_{m,3}$ by simple operations. These properties can be preserved by the process of distribution splitting. Moreover, the process of distribution splitting is non-trivial only for $D_{m,k}(x),\ k=2,3$. This corresponds to the assertion that one needs only two renormalization constants for the Wess–Zumino model; see [22,15].

Now we have the following.

Theorem 3.3. In the conditions of the preceding proposition, the second order chronological product $T(x,y)$ can be chosen such that it verifies

$$[Q,T(x,y)] = {\rm i}\frac{\partial}{\partial x^\mu}T_1^\mu(x,y) + {\rm i}\frac{\partial}{\partial y^\mu}T_2^\mu(x,y) \tag{3.2.4}$$

for some associated chronological products $T_i^\mu(x,y),\ i=1,2$ if one takes in (3.2.2):

$$N(x) \equiv {\rm i} : A(x)^4 : +{\rm i} : B(x)^4 : +2{\rm i} : A(x)^2B(x)^2 : . \tag{3.2.5}$$

Proof. We follow the model of [11] and compute the commutators:

$$D_1^\mu(x,y) \equiv [T_1^\mu(x), T_1(y)], \quad D_2^\mu(x,y) = D_1^\mu(y,x). \tag{3.2.6}$$

By direct computation we have

$$\begin{aligned}[][T_1^\mu(x),T_1(y)] = &-2{\rm i} : A(x)^2\gamma^\mu S_m(x-y)\psi(y)A(y) :\\ &-2 : A(x)^2\gamma^\mu S_m(x-y)\gamma_5\psi(y)B(y) :\\ &+2{\rm i} : B(x)^2\gamma^\mu S_m(x-y)\gamma_5\psi(y)B(y) :\\ &+2 : B(x)^2\gamma^\mu S_m(x-y)\psi(y)A(y) :\\ &+4 : A(x)B(x)\gamma_5\gamma^\mu S_m(x-y)\psi(y)A(y) :\\ &-4{\rm i} : A(x)B(x)\gamma_5\gamma^\mu S_m(x-y)\psi(y)B(y) : +\cdots,\end{aligned} \tag{3.2.7}$$

where the expressions $\cdots$ cannot produce anomalies.

We perform the canonical causal splitting of the expression $(\partial/\partial x^\mu)D_1^\mu(x,y)$ and obtain the usual delta-distribution anomaly:

$$\begin{aligned}A_1(x,y) \equiv\ & 2\delta(x-y)\\ &\times[-{\rm i} : A(x)^3\psi(x) : + : B(x)^3\gamma_5\psi(x) :\\ &+ : A(x)^2B(x)^2\gamma_5\psi(x) :\\ &-{\rm i} : A(x)B(x)^2\psi(x) :],\end{aligned} \tag{3.2.8}$$

and a similar contribution follows from the other commutator. But one easily proves that

$$A_1(x,y) = \frac{1}{2}[Q, : A(x)^4 : + : B(x)^4 : +2 : A(x)^2B(x)^2 :], \tag{3.2.9}$$

so the “anomalies” can be eliminated by a proper choice of the finite renormalization $N(x)$.

We remark that, quite similarly to the case of Yang–Mills theories, we have obtained the second order contribution of the usual Wess–Zumino Lagrangian from the formulation without the supplementary fields [22]. However, in this case, the anomalies can be completely eliminated by a proper choice of the finite renormalization. Moreover, the arbitrariness of $T(x,y)$ is of the form $\delta(x-y)\times$(3.1.28) if one requires that the canonical dimension does not exceed 4. This is again consistent with the assertion from the traditional approaches to renormalization theory. In this model, at least up to order 2, one needs to renormalize only two constants: the mass and the overall coupling constant.

4 Ward identities and anomalies

4.1 The main theorem

We consider the Wess–Zumino model as defined by the Lagrangian (3.1.28) and show that we can implement supersymmetry invariance in all orders of perturbation theory.

Theorem 4.1. One can construct the chronological products $T(X)$ such that, beside Bogoliubov axioms, the following relation is valid:

$$[Q,T(X)] = {\rm i}\sum_{l=1}^n \frac{\partial}{\partial x_l^\mu}T_l^\mu(X), \quad \forall|X|, \tag{4.1.1}$$

where $T_l^\mu(X)$ are some auxiliary chronological products which can be chosen such that

$$T_l^\mu(X)^c = T_l^\mu(X), \quad \forall|X|. \tag{4.1.2}$$

Proof. The main trick is to formulate carefully the *induction hypothesis*. We suppose that we have constructed the chronological products $T(x_1,\cdots,x_p)$, $p=1,\ldots,n-1$ having the following properties: (2.1.2)–(2.1.4) and (2.1.6) for $|X|\leq n-1$. We also suppose that we have a more precise form of the Wick expansion property:

$$T(X)=\sum_{|I|=|J|}:\prod_{i\in I}\bar{\psi}_{\alpha_i}(x_i)t_{I,J,K,P}(X)_{\alpha_I;\beta_J}\prod_{j\in J}\psi_{\beta_i}(x_j)\times\prod_{k\in K}A(x_k)\prod_{p\in P}B(x_p): \tag{4.1.3}$$

where

(a) the sum runs over all distinct triplets $I,J,K,P\subset\{1,\ldots,n-1\}$;
(b) we have denoted $\alpha_I\equiv\{\alpha_i\}_{i\in I}$ and $\beta_J\equiv\{\beta_j\}_{j\in J}$;
(c) the expressions $t_{I,J,K,P}(X)$ are numerical distributions (renormalized Feynman amplitudes); more precisely, they take values in the matrix space $M_{\mathbb{C}}(4,4)^{\otimes|I|}$;
(d) they are $SL(2,\mathbb{C})$-covariant such that we have (2.1.3);
(e) we can suppose convenient (anti-) symmetry properties of the numerical distributions without losing generality;
(f) we have the limitation

$$\omega(t_{I,J,K,P})\leq 4-3|I|-|K|-|L|. \tag{4.1.4}$$

We note that in this case we also have (2.2.6) for $|X_1|+|X_2|\leq n-1$.

We also suppose that we have constructed the Wick polynomials $T^\mu_l(X),\ |X\leq n-1$ such that we have properties analogous to (2.1.2), (2.1.4), and (4.1.3). We use the convention

$$T(\emptyset)\equiv\mathbf{1},\quad T^\mu_l(\emptyset)\equiv 0,\quad T^\mu_l(X)\equiv 0,\quad \text{for } l\notin X. \tag{4.1.5}$$

Then the induction hypothesis is supplemented as follows.

(1) Symmetry:

$$T^\mu_{\pi^{-1}(l)}(x_{\pi(1)},\cdots x_{\pi(p)})=T^\mu_l(x_1,\cdots x_p),\quad \forall\pi\in\mathcal{P}_p, \tag{4.1.6}$$

for $p=1,\ldots,n-1$;

(2) Covariance with respect to $SL(2,\mathbb{C})$:

$$U_{a,A}T^\mu_l(x_1,\cdots,x_p)U^{-1}_{a,A}=\delta(A^{-1})^\mu{}_\rho T^\rho_l(\delta(A)\cdot x_1+a,\cdots,\delta(A)\cdot x_p+a), \tag{4.1.7}$$

$p=1,\ldots,n-1$;

(3) Charge conjugation invariance:

$$U_C T^\mu_l(X)U_C^{-1}=T^\mu_l(X),\quad |X|\leq n-1\Longleftrightarrow T^\mu_l(X)^c=T^\mu_l(X),\quad |X|\leq n-1. \tag{4.1.8}$$

(4) Causality

$$T^\mu_l(X_1X_2)=T^\mu_l(X_1)T(X_2)+T(X_1)T^\mu_l(X_2)\quad \forall X_1\geq X_2,\quad |X_1|+|X_2|\leq n-1. \tag{4.1.9}$$

(5) Wick expansion property:

$$T^\mu_l(X)_\epsilon=\sum_{|J|=|I|+1}:\prod_{i\in I}\bar{\psi}_{\alpha_i}(x_i)t^\mu_{l;I,J,K,P}(X)_{\epsilon\alpha_I;\beta_J}\times\prod_{j\in J}\psi_{\beta_i}(x_j)\prod_{k\in K}A(x_k)\times\prod_{p\in P}B(x_p): \tag{4.1.10}$$

where the sum runs over all distinct triplets $I,J,K,P\subset\{1,\ldots,n-1\}$ verifying $|J|=|I|+1$; the expressions $t^\mu_{l;I,J,K,P}$ are numerical distributions taking values in the matrix space $M_{\mathbb{C}}(4,4)^{\otimes|J|}$, they are $SL(2,\mathbb{C})$-covariant and have convenient (anti-) symmetry properties. Moreover, we make the inductive hypothesis:

$$\omega(t^\mu_{l;I,J,K,P})\leq 3-3|I|-|K|-|L|. \tag{4.1.11}$$

We note that in this case we also have

$$[T^{\mu_1}_{l_1}(X_1),T^{\mu_2}_{l_2}(X_2)]=0,\quad [T^\mu_l(X_1),T(X_2)]=0,\quad \text{if } X_1\sim X_2, \tag{4.1.12}$$

for $|X_1|+|X_2|\leq n-1$.

(6) Supersymmetry invariance: we require that we have (4.1.1) for $|X|\leq n-1$.
(7) In the case $J=lJ'$ the distribution $t^\mu_{l;I,J,K,P}(X)$ is "proportional" to γ^μ i.e. we have

$$t^\mu_{l;I,J,K,P}(X)=t_{l;I,J,K,P}(X)\otimes\gamma^\mu. \tag{4.1.13}$$

The corresponding Feynman graphs are 1-particle reducible.

Let us note that, in the formulation of [17,5], we have

$$T(x_1,\cdots,x_n)\equiv T(T(x_1),\ldots,T(x_n)), \tag{4.1.14}$$
$$T^\mu_l(x_1,\cdots,x_n)\equiv T(T(x_1),\ldots,T^\mu(x_l),\ldots,T(x_n)).$$

We observe that the induction hypothesis is valid for $|X|=1$ according to the preceding section. We suppose that it is true for $|X|\leq n-1$ and prove it for $|X|=n$.

Now we can proceed in strict analogy with Sect. 2.2. The proof of the following items below goes in strict analogy to the proof of the similar statements from the previous subsection and can easily be provided with minimal modifications.

One constructs from $T(X),\ T^\mu_l(X),\ |X|\leq n-1$ the expressions $\bar{T}(X),\ |X|\leq n-1$ according to (2.1.5) and similarly $\bar{T}^\mu_l(X),\ |X|\leq n-1$, according to

$$(-1)^{|X|}\bar{T}^\mu_l(X)\equiv\sum_{r=1}^{|X|}(-1)^r\sum_{X_1,\cdots,X_r\in\mathrm{Part}(X)}\times[T^\mu_l(X_1)T(X_2)\cdots T(X_r)+\cdots+T(X_1)\cdots T(X_{r-1})T^\mu_l(X_r)]; \tag{4.1.15}$$

we use in an essential way the convention (4.1.5). Next, we define in analogy to (2.2.7) and (2.2.8) the following expressions for $|X|=n$:

$$A'^{\mu}_l(X) \equiv \sum_{\substack{X_1,X_2\in \mathrm{Part}(X)\\ X_2\neq\emptyset, x_n\in X_1}} \left[T^{\mu}_l(X_1)\bar{T}(X_2) + T(X_1)\bar{T}^{\mu}_l(X_2)\right], \tag{4.1.16}$$

$$R'^{\mu}_l(X) \equiv \sum_{\substack{X_1,X_2\in \mathrm{Part}(X)\\ X_2\neq\emptyset, x_n\in X_1}} \left[\bar{T}^{\mu}_l(X_1)T(X_2) + \bar{T}(X_1)T^{\mu}_l(X_2)\right]. \tag{4.1.17}$$

Next, we define in analogy to (2.2.9) the expression

$$D^{\mu}_n(X) \equiv A'^{\mu}_l(X) - R'^{\mu}_l(X), \tag{4.1.18}$$

and prove that it has causal support, i.e., $\mathrm{supp}(D^{\mu}_n(x_1,\dots,x_{n-1};x_n)) \subset \Gamma^+(x_n)\cup\Gamma^-(x_n)$. The proof is completely analogous to the standard proof from [8].

From the Wick expansion properties (4.1.3) and (4.1.10) we also have, with the same conventions,

$$D(X) = \sum_{|I|=|J|} :\prod_{i\in I}\bar{\psi}_{\alpha_i}(x_i) d_{I,J,K,P}(X)_{\alpha_I;\beta_J} \times \prod_{j\in J}\psi_{\beta_i}(x_j)\prod_{k\in K}A(x_k)\prod_{p\in P}B(x_p): \tag{4.1.19}$$

$$D^{\mu}_l(X)_\epsilon = \sum_{|J|=|I|+1} :\prod_{i\in I}\bar{\psi}_{\alpha_i}(x_i) d^{\mu}_{l;I,J,K,P}(X)_{\epsilon\alpha_I;\beta_J} \times \prod_{j\in J}\psi_{\beta_i}(x_j)\prod_{k\in K}A(x_k)\prod_{p\in P}B(x_p): \tag{4.1.20}$$

where $d_{\dots}^{\dots}(X)$ are numerical distributions verifying the following properties:

(a) $SL(2,\mathbb{C})$-covariance;
(b) causal support, i.e, $\mathrm{supp}(d_{\dots}^{\dots}(x_1,\dots,x_{n-1};x_n)) \subset \Gamma^+(x_n)\cup\Gamma^-(x_n)$;
(c) limitation on the order of singularity:

$$\omega(d_{I,J,K,P}) \leq 4 - 3|I| - |K| - |L|,$$
$$\omega(d^{\mu}_{l;I,J,K,P}) \leq 3 - 3|I| - |K| - |L|. \tag{4.1.21}$$

The absence of a derivative in the Wick monomials $W_i(X)$ is again essential in establishing these relations.

As a consequence, there exists a $SL(2,\mathbb{C})$-covariant causal splitting:

$$D^{\mu}_l(X) = A^{\mu}_l(X) - R^{\mu}_l(X), \quad |X| = n, \tag{4.1.22}$$

with $\mathrm{supp}(A^{\mu}_l(X)) \subset \Gamma^+(x_n)$ and $\mathrm{supp}(R^{\mu}_l(X)) \subset \Gamma^-(x_n)$ for all $l = 1,\dots,n$.

We also have from the induction hypothesis in analogy with (2.2.18):

$$D^{\mu}_l(X)^c = (-1)^{n-1}D^{\mu}_l(X), \quad |X| = n. \tag{4.1.23}$$

Now we investigate the possible obstruction to the extension of the identity (4.1.1) for $|X| = n$. We first prove by direct computation that we have

$$[Q, D(X)] = \mathrm{i}\sum_{l\in X}\frac{\partial}{\partial x^{\mu}_l}D^{\mu}_l(X), \quad |X| = n. \tag{4.1.24}$$

We substitute here the causal decompositions (2.2.17) and (4.1.22) in the preceding relation and we get

$$[Q, A(X)] - \mathrm{i}\sum_{l=1}^{n}\frac{\partial}{\partial x^{\mu}_l}A^{\mu}_l(X) = [Q, R(X)] - \mathrm{i}\sum_{l=1}^{n}\frac{\partial}{\partial x^{\mu}_l}R^{\mu}_l(X). \tag{4.1.25}$$

Now the left hand side has support in $\Gamma^+(x_n)$ and the right hand side in $\Gamma^-(x_n)$ so the common value, denoted by $P(X)$ should have the support in $\Gamma^+(x_n)\cap\Gamma^-(x_n) = \{x_1 = \dots = x_n\}$. This means that we have

$$[Q, A(X)] - \mathrm{i}\sum_{l=1}^{n}\frac{\partial}{\partial x^{\mu}_l}A^{\mu}_l(X) = P(X), \tag{4.1.26}$$

where $P(X)$ has the structure

$$P(X) = \sum_i \left[p_i(\partial)\delta^{n-1}(X)\right] W_i(x); \tag{4.1.27}$$

here p_i are polynomials in the derivatives with the maximal degree restricted by

$$\deg(p_i) + \deg(W_i) \leq 5 \tag{4.1.28}$$

and

$$\delta^{n-1}(X) \equiv \delta(x_1 - x_n)\cdots\delta(x_{n-1} - x_n). \tag{4.1.29}$$

It is easy to see that the "anomaly" can be produced only by those terms appearing in the Wick expansions of $D(X)$ and $D^{\mu}_l(X)$ for which the Wick monomials are restricted by $\omega(W_i) \leq 5$. We will show in the next subsections that one can choose $P(X) = 0$. We will write such a generic form of these terms from $D(X)$ and $D^{\mu}_l(X)$ in the next two subsections.

4.2 The expression of $D(X)$

The terms corresponding to canonical dimension ≤ 5 from (4.1.19) are

A. $\omega(W_K) = 1$

$$D^{(1)}(X) = \sum d^{(1)}_i(X)A(x_i), \tag{4.2.1}$$
$$D^{(2)}(X) = \sum d^{(2)}_i(X)B(x_i). \tag{4.2.2}$$

B. $\omega(W_K) = 2$

$$D^{(3)}(X) = \sum d^{(3)}_{ij}(X) : A(x_i)A(x_j): \tag{4.2.3}$$
$$D^{(4)}(X) = \sum d^{(4)}_{ij}(X) : A(x_i)B(x_j): \tag{4.2.4}$$
$$D^{(5)}(X) = \sum d^{(5)}_{ij}(X) : B(x_i)B(x_j): \tag{4.2.5}$$

C. $\omega(W_K) = 3$

$$D^{(6)}(X) = \sum d^{(6)}_{ijk}(X) : A(x_i)A(x_j)A(x_k) : \quad (4.2.6)$$

$$D^{(7)}(X) = \sum d^{(7)}_{ijk}(X) : A(x_i)A(x_j)B(x_k) : \quad (4.2.7)$$

$$D^{(8)}(X) = \sum d^{(8)}_{ijk}(X) : A(x_i)B(x_j)B(x_k) : \quad (4.2.8)$$

$$D^{(9)}(X) = \sum d^{(9)}_{ijk}(X) : B(x_i)B(x_j)B(x_k) : \quad (4.2.9)$$

$$D^{(10)}(X) = \sum : \overline{\psi}(x_i) d^{(10)}_{ij}(X) \psi(x_j) : \quad (4.2.10)$$

D. $\omega(W_K) = 4$

$$D^{(11)}(X) = \sum d^{(11)}_{ijkp}(X) : A(x_i)A(x_j)A(x_k)A(x_p) : \quad (4.2.11)$$

$$D^{(12)}(X) = \sum d^{(12)}_{ijkp}(X) : A(x_i)A(x_j)A(x_k)B(x_p) : \quad (4.2.12)$$

$$D^{(13)}(X) = \sum d^{(13)}_{ijkp}(X) : A(x_i)A(x_j)B(x_k)B(x_p) : \quad (4.2.13)$$

$$D^{(14)}(X) = \sum d^{(14)}_{ijkp}(X) : A(x_i)B(x_j)B(x_k)B(x_p) : \quad (4.2.14)$$

$$D^{(15)}(X) = \sum d^{(15)}_{ijkp}(X) : B(x_i)B(x_j)B(x_k)B(x_p) : \quad (4.2.15)$$

$$D^{(16)}(X) = \sum : \overline{\psi}(x_i) d^{(16)}_{ijk}(X) \psi(x_j)A(x_k) : \quad (4.2.16)$$

$$D^{(17)}(X) = \sum : \overline{\psi}(x_i) d^{(17)}_{ijk}(X) \psi(x_j)B(x_k) : \quad (4.2.17)$$

E. $\omega(W_K) = 5$

$$D^{(18)}(X) = \sum d^{(18)}_{ijkpq}(X) : A(x_i)A(x_j)A(x_k)A(x_p)A(x_q) : \quad (4.2.18)$$

$$D^{(19)}(X) = \sum d^{(19)}_{ijkpq}(X) : A(x_i)A(x_j)A(x_k)A(x_p)B(x_q) : \quad (4.2.19)$$

$$D^{(20)}(X) = \sum d^{(20)}_{ijkpq}(X) : A(x_i)A(x_j)A(x_k)B(x_p)B(x_q) : \quad (4.2.20)$$

$$D^{(21)}(X) = \sum d^{(21)}_{ijkpq}(X) : A(x_i)A(x_j)B(x_k)B(x_p)B(x_q) : \quad (4.2.21)$$

$$D^{(22)}(X) = \sum d^{(22)}_{ijkpq}(X) : A(x_i)B(x_j)B(x_k)B(x_p)B(x_q) : \quad (4.2.22)$$

$$D^{(23)}(X) = \sum d^{(23)}_{ijkpq}(X) : B(x_i)B(x_j)B(x_k)B(x_p)B(x_q) : \quad (4.2.23)$$

$$D^{(24)}(X) = \sum : \overline{\psi}(x_i) d^{(24)}_{ijkp}(X) \psi(x_j)A(x_k)A(x_p) : \quad (4.2.24)$$

$$D^{(25)}(X) = \sum : \overline{\psi}(x_i) d^{(25)}_{ijkp}(X) \psi(x_j)A(x_k)B(x_p) : \quad (4.2.25)$$

$$D^{(26)}(X) = \sum : \overline{\psi}(x_i) d^{(26)}_{ijkp}(X) \psi(x_j)B(x_k)B(x_p) : \quad (4.2.26)$$

The term proportional to the identity operator **1** is omitted because it does not contribute to (4.1.24). We assume that $d^{(10)}, d^{(16)}, d^{(17)}, d^{(24)}$–$d^{(26)}$ are matrix-valued distributions; more precisely that they have values in $M_{\mathbb{C}}(4,4)$. Moreover, it can be proved that these distribution can be chosen such that they verify

$$C^{-1} d^{(\alpha)}_{ij\dots}(X) C = -d^{(\alpha)}_{ji\dots}(\pi_{ij}(X))^T \quad (4.2.27)$$

without losing generality. The other expressions $d^{(\alpha)}$ are numerical distributions. The distributions $d^{(\alpha)}$, $\alpha = 1,\dots, 26$ are $SL(2,\mathbb{C})$-covariant and have causal support.

4.3 The expression of $D^{\mu}_l(X)$

The terms corresponding to canonical dimension ≤ 5 from (4.1.20) are

A. $\omega(W_K) = 3/2$

$$D^{(1)\mu}_l(X) = \sum d^{(1)\mu}_{l;i}(X) \psi(x_i) \quad (4.3.1)$$

B. $\omega(W_K) = 5/2$

$$D^{(2)\mu}_l(X) = \sum d^{(2)\mu}_{l;ij}(X) : \psi(x_i)A(x_j) : \quad (4.3.2)$$

$$D^{(3)\mu}_l(X) = \sum d^{(3)\mu}_{l;ij}(X) : \psi(x_i)B(x_j) : \quad (4.3.3)$$

C. $\omega(W_K) = 7/2$

$$D^{(4)\mu}_l(X) = \sum d^{(4)\mu}_{l;ijk}(X) : \psi(x_i)A(x_j)A(x_k) : \quad (4.3.4)$$

$$D^{(5)\mu}_l(X) = \sum d^{(5)\mu}_{l;ijk}(X) : \psi(x_i)A(x_j)B(x_k) : \quad (4.3.5)$$

$$D^{(6)\mu}_l(X) = \sum d^{(6)\mu}_{l;ijk}(X) : \psi(x_i)B(x_j)B(x_k) : \quad (4.3.6)$$

D. $\omega(W_K) = 9/2$

$$D^{(7)\mu}_l(X) = \sum d^{(7)\mu}_{l;ijkp}(X) : \psi(x_i)A(x_j)A(x_k)A(x_p) : \quad (4.3.7)$$

$$D^{(8)\mu}_l(X) = \sum d^{(8)\mu}_{l;ijkp}(X) : \psi(x_i)A(x_j)A(x_k)B(x_p) : \quad (4.3.8)$$

$$D^{(9)\mu}_l(X) = \sum d^{(9)\mu}_{l;ijkp}(X) : \psi(x_i)A(x_j)B(x_k)B(x_p) : \quad (4.3.9)$$

$$D^{(10)\mu}_l(X) = \sum d^{(10)\mu}_{l;ijkp}(X) : \psi(x_i)B(x_j)B(x_k)B(x_p) : \quad (4.3.10)$$

$$D^{(11)\mu}_l(X) = \sum : \overline{\psi(x_i)} d^{(11)\mu}_{l;ijk}(X) \psi(x_j)\psi(x_k) : \quad (4.3.11)$$

We assume that they are matrix-valued distributions: $d^{(1)\mu}_{l;I}$–$d^{(10)\mu}_{l;I} \in M_{\mathbb{C}}(4,4)$ and $d^{(11)\mu}_{l;ijk} \in M_{\mathbb{C}}(4,4)^{\otimes 2}$; we can impose for these distributions the condition

$$C^{-1} d^{(\alpha)}_{l;I}(X) C = -d^{(\alpha)}_{l;I}(X)^T, \quad \alpha = 1,\dots,10, \quad (4.3.12)$$

and

$$C^{-1} d^{(11)}_{l;ijk}(X) C = -d^{(11)}_{l;jik}(\pi_{ij}(X))^T \tag{4.3.13}$$

without losing generality. The distributions $d^{(\alpha)\mu}$, $\alpha = 1, \ldots, 11$ are $SL(2,\mathbb{C})$-covariant and have causal support.

Moreover, we have from the induction hypothesis (4.1.13) that

$$d^{(\alpha)}_{l;lI} = d^{(\alpha)}_{l;lI} \gamma^\mu, \quad \alpha = 1, \ldots, 10, \tag{4.3.14}$$

where $d^{(\alpha)}_{i;I}$ are numerical distribution, and

$$d^{(11)}_{j;ijk} = d^{(11)}_{j;ijk} \otimes \gamma^\mu, \tag{4.3.15}$$

where $d^{(11)}_{j;ijk}$ is a matrix-valued distribution, more precisely one with values in $M_{\mathbb{C}}(4,4)$.

4.4 The basic equations

The expression $\mathrm{i}[Q, D(X)] + \sum_l (\partial/\partial x^\mu_l) D^\mu_l(X)$ is a Wick sum and the relevant contributions following from the preceding two subsections are

1.1 The coefficient of the monomial $\psi(x_i)$:

$$d^{(1)}_i(X) + \mathrm{i} d^{(2)}_i(X)\gamma_5 - \mathrm{i} m d^{(1)}_{i;i}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(1)\mu}_{l;i}(X). \tag{4.4.1}$$

B. $\omega(W_K) = 5/2$

2.1 The coefficient of the monomial $:\psi(x_i)A(x_j):$

$$2d^{(3)}_{ij}(X) + \mathrm{i} d^{(4)}_{ij}(X)\gamma_5 - 2m d^{(10)}_{ji}(X') - \mathrm{i} m d^{(2)}_{i;ij}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(2)\mu}_{l;ij}(X) \tag{4.4.2}$$

2.2 The coefficient of the monomial $:\psi(x_i)B(x_j):$

$$d^{(4)}_{ij}(X) + 2\mathrm{i} d^{(5)}_{ij}(X)\gamma_5 - 2\mathrm{i} m \gamma_5 d^{(10)}_{ji}(X') - \mathrm{i} m d^{(3)}_{i;ij}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(3)\mu}_{l;ij}(X) \tag{4.4.3}$$

C. $\omega(W_K) = 7/2$

3.1 The coefficient of the monomial $:\psi(x_i)A(x_j)A(x_k):$

$$3d^{(6)}_{ijk}(X) + \mathrm{i} d^{(7)}_{jki}(X')\gamma_5 - 2m d^{(16)}_{jik}(X') - \mathrm{i} m d^{(4)}_{i;ijk}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(4)\mu}_{l;ijk}(X). \tag{4.4.4}$$

3.2 The coefficient of the monomial $:\psi(x_i)A(x_j)B(x_k):$

$$2d^{(7)}_{ijk}(X) + 2\mathrm{i} d^{(8)}_{jki;abc}(X')\gamma_5 - 2\mathrm{i} m d^{(16)}_{kij}(X')\gamma_5 - 2m d^{(17)}_{jik}(X') - \mathrm{i} m d^{(5)}_{i;ijk}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(5)\mu}_{l;ijk}(X). \tag{4.4.5}$$

3.3 The coefficient of the monomial $:\psi(x_i)B(x_j)B(x_k):$

$$d^{(8)}_{ijk}(X) + 3\mathrm{i} d^{(9)}_{ijk}(X)\gamma_5 - 2\mathrm{i} m d^{(17)}_{jik}(X')\gamma_5 - \mathrm{i} m d^{(6)}_{i;ijk}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(6)\mu}_{l;ijk}(X). \tag{4.4.6}$$

3.4 The coefficient of the monomial $:\partial_\mu A(x_j)\psi(x_j):$

$$2\mathrm{i}\gamma^\mu d^{(10)}_{ij}(X) + d^{(2)\mu}_{i;ji}(X'). \tag{4.4.7}$$

3.5 The coefficient of the monomial $:\partial_\mu B(x_i)\psi(x_j):$

$$-2\gamma_5\gamma^\mu d^{(10)}_{ij}(X) + d^{(3)\mu}_{i;ji}(X'). \tag{4.4.8}$$

D. $\omega(W_K) = 9/2$

4.1 The coefficient of the monomial $:\psi(x_i)A(x_j)A(x_k)A(x_p):$

$$4d^{(11)}_{ijkp}(X) + \mathrm{i} d^{(12)}_{pijk}(X')\gamma_5 - 2m d^{(24)}_{jikp}(X') - \mathrm{i} m d^{(7)}_{i;ijkp}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(7)\mu}_{l;ijkp}(X). \tag{4.4.9}$$

4.2 The coefficient of the monomial $:\psi(x_i)A(x_j)A(x_k)B(x_p):$

$$3d^{(12)}_{ijkp}(X) + 2\mathrm{i} d^{(13)}_{kijp}(X')\gamma_5 - 2\mathrm{i} m d^{(24)}_{pijk}(X')\gamma_5 - 2m d^{(25)}_{jikp}(X') - \mathrm{i} m d^{(8)}_{i;ijkp}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(8)\mu}_{l;ijkp}(X). \tag{4.4.10}$$

4.3 The coefficient of the monomial $:\psi(x_i)A(x_j)B(x_k)B(x_p):$

$$2d^{(13)}_{ijkp}(X) + 3\mathrm{i} d^{(14)}_{jikp}(X')\gamma_5 - 2\mathrm{i} m d^{(25)}_{kij}(X')\gamma_5 - 2m d^{(26)}_{jikp}(X') - \mathrm{i} m d^{(9)}_{i;ijkp}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(9)\mu}_{l;ijkp}(X). \tag{4.4.11}$$

4.4 The coefficient of the monomial $:\psi(x_i)B(x_j)B(x_k)B(x_p):$

$$d^{(14)}_{ijkp}(X) + 4\mathrm{i} d^{(15)}_{ijkp}(X)\gamma_5 - 2\mathrm{i} m d^{(26)}_{jikp}(X')\gamma_5 - \mathrm{i} m d^{(10)}_{i;ijkp}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(10)\mu}_{l;ijkp}(X). \tag{4.4.12}$$

4.5 The coefficient of the monomial $:\psi(x_i)\partial_\mu A(x_j)A(x_k):$

$$2\mathrm{i}\gamma^\mu d^{(16)}_{jik}(X') + d^{(4)\mu}_{j;ijk}(X). \tag{4.4.13}$$

4.6 The coefficient of the monomial $:\psi(x_i)\partial_\mu A(x_j)B(x_k):$

$$2\mathrm{i}\gamma^\mu d^{(17)\mu}_{jik}(X') + d^{(5)\mu}_{j;ijk}(X). \tag{4.4.14}$$

4.7 The coefficient of the monomial $:\psi(x_i)A(x_j)\partial_\mu B(x_k):$

$$-2\gamma_5\gamma^\mu d^{(16)}_{kij}(X') + d^{(5)\mu}_{k;ijk}(X). \tag{4.4.15}$$

4.8 The coefficient of the monomial $:\psi(x_i)\partial_\mu B(x_j)B(x_k):$

$$-\gamma_5\gamma^\mu d^{(17)}_{jik}(X') + d^{(6)\mu}_{j;ijk}(X). \tag{4.4.16}$$

4.9 The coefficient of the monomial $:\overline{\psi}(x_i)\psi(x_j)\psi(x_k):$

$$d^{(16)}_{ijk}(X)\otimes\mathbf{1} + \mathrm{i}d^{(17)}_{ijk}(X)\otimes\gamma_5 - 3\mathrm{i}md^{(11)}_{k;ijk}(X) + \sum_l \frac{\partial}{\partial x^\mu_l} d^{(11)\mu}_{l;ijk}(X). \tag{4.4.17}$$

In these equations we mean by X' the corresponding permutation of the variables X. The expression of the anomaly $P(X)$ is given by (4.1.27) where only the 17 Wick monomials listed above can appear.

All anomalies p_i can be eliminated purely algebraically by redefining some of the causal splittings. First, we take a causal splitting of all distributions

$$d^{(\alpha)} = a^{(\alpha)} - r^{(\alpha)}, \quad d^{(\alpha)\mu} = a^{(\alpha)\mu} - r^{(\alpha)\mu}, \tag{4.4.18}$$

verifying $SL(2,\mathbb{C})$-covariance and preserving the order of singularity. (We make the labelling in such a way that for $\alpha = 1,\ldots,26$ and $\alpha = 1,\ldots,11$ respectively we have the distributions from the preceding two subsections.) The expressions $A(X)$ and $A^\mu_l(X)$ are defined according to the relations of the type (2.2.4).

Then we notice that we can absorb all anomalies in

$$\begin{aligned}&a^{(1)}_i(X), a^{(3)}_{ij}(X), a^{(4)}_{ij}(X), a^{(6)}_{ijk}(X), a^{(7)}_{ijk}(X), a^{(8)}_{ijk}(X),\\ &a^{(2)\mu}_{j;ji}(X), a^{(3)\mu}_{j;ji}(X), a^{(11)}_{ijkp}(X),\\ &a^{(12)}_{ijkp}(X), a^{(13)}_{ijkp}(X), a^{(14)}_{ijkp}(X), a^{(4)\mu}_{j;ijk}(X), a^{(5)\mu}_{j;ijk}(X),\\ &a^{(5)\mu}_{k;ijk}(X), a^{(6)\mu}_{j;ijk}(X), a^{(16)}_{ijk}(X)\end{aligned} \tag{4.4.19}$$

respectively.

This means that one can make the causal splitting such that (4.1.26) is

$$[Q, A(X)] = \mathrm{i}\sum_{l=1}^n \frac{\partial}{\partial x^\mu_l} A^\mu_l(X). \tag{4.4.20}$$

From the relation (4.4.20) one can obtain by Hermitian conjugation

$$[Q, A(X)^*] = \mathrm{i}\sum_{l=1}^n \frac{\partial}{\partial x^\mu_l} A^\mu_l(X)^c, \tag{4.4.21}$$

where use has been made of the relation (3.1.23), which says that the supercharge is a Majorana spinor, and of the relations (2.2.18) and (4.1.23). If one makes the substitutions (2.2.20) and

$$A^\mu_l(X) \to \frac{1}{2}\left[A^\mu_l(X) + (-1)^{n-1}(A(X)^\mu_l)^c\right], \tag{4.4.22}$$

we still have a legitimate causal decomposition of the type (2.2.17) and (4.1.22); the consistency is ensured by (4.1.23). It follows that the causal splitting can be chosen such that

$$A(X)^* = (-1)^{n-1}A(X), \quad A^\mu_l(X)^c = (-1)^{n-1}A^\mu_l(X). \tag{4.4.23}$$

This relation is essential in establishing the unitarity axiom in order n (see [8]).

Finally we define the chronological products as in Sect. 2.2 and by analogy

$$T^\mu_l(X) \equiv A^\mu_l(X) - A'^\mu_l(X) = R^\mu_l(X) - R'^\mu_l(X). \tag{4.4.24}$$

These expressions satisfy Poincaré covariance, causality and unitarity conditions. If we make the substitution (2.2.22) and analogously

$$T^\mu_l(x_1,\cdots,x_n) \to \frac{1}{n!}\sum_\pi T^\mu_{\pi^{-1}(l)}(x_{\pi(1)},\cdots,x_{\pi(n)}), \tag{4.4.25}$$

then we also have the symmetry axioms (2.1.2) and (4.1.6). Finally, one can prove that the rest of the induction hypothesis is true in order n of the perturbation theory. Some effort is required for (4.1.13).

The invariance of the S-matrix with respect to space-time inversions can be obtained as in the case of quantum electrodynamics [16].

4.5 The conservation of the supercurrent

The analysis from [14] is based on the conservation of the supercurrent. We give below the analysis in the first order and compare with the result of [14].

First we have the following.

Proposition 4.2. The following relation is true:

$$\begin{aligned}[J^\mu(x), T(y)] &= D_m(x-y)A^\mu(x,y)\\ &+\frac{\partial}{\partial x^\nu}D_m(x-y)\gamma^\nu\gamma^\mu A(x,y)\\ &+2m[\gamma\cdot\partial,\gamma^\mu]D_{m,2}(x-y)\\ &+2:\partial_\nu A(x)\gamma^\nu\gamma^\mu S_m(x-y)A(y)\psi(y):\\ &-2\mathrm{i}:\partial_\nu A(x)\gamma^\nu\gamma^\mu S_m(x-y)B(y)\gamma_5\psi(y):\\ &+2\mathrm{i}:\partial_\nu B(x)\gamma_5\gamma^\nu\gamma^\mu S_m(x-y)A(y)\psi(y):\\ &+2:\partial_\nu B(x)\gamma_5\gamma^\nu\gamma^\mu S_m(x-y)A(y)\gamma_5\psi(y):\\ &+2\mathrm{i}m:A(x)\gamma^\mu S_m(x-y)A(y)\psi(y):\\ &+2m:A(x)\gamma^\mu S_m(x-y)B(y)\gamma_5\psi(y):\\ &-2m:B(x)\gamma_5\gamma^\mu S_m(x-y)A(y)\psi(y):\\ &+2\mathrm{i}m:B(x)\gamma_5\gamma^\mu S_m(x-y)B(y)\gamma_5\psi(y):\end{aligned} \tag{4.5.1}$$

where

$$\begin{aligned}A^\mu(x,y) \equiv\ &3\mathrm{i}m^2:\gamma^\mu\psi(x)A(y)^2:\\ &+\mathrm{i}m^2:\gamma^\mu\psi(x)B(y)^2:\\ &+2m^2:\gamma_5\gamma^\mu\psi(x)A(y)B(y):\\ &+\mathrm{i}m:\gamma^\mu\psi(x)\overline{\psi}(y)\psi(y):\\ &+\mathrm{i}m:\gamma_5\gamma^\mu\psi(x)\overline{\psi}(y)\gamma_5\psi(y):\end{aligned} \tag{4.5.2}$$

and

$$A(x,y) \equiv 3m : \psi(x)A(y)^2 : +m : \psi(x)B(y)^2 : \\ +2im : \psi(x)A(y)B(y) : +m : \psi(x)\overline{\psi}(y)\psi(y) : \\ +m : \gamma_5\psi(x)\overline{\psi}(y)\gamma_5\psi(y) : \qquad (4.5.3)$$

Following the usual procedure of constructing the chronological products of the general type $T(A_1(x_1), \ldots, A_n(x_n))$ (see [17], [5]) one can find that, up to finite renormalizations, one can take

$$T(J^\mu(x), T(y)) = D^{\mathrm{F}}_m(x-y)A^\mu(x,y) \\ +\frac{\partial}{\partial x^\nu}D^{\mathrm{F}}_m(x-y)\gamma^\nu\gamma^\mu A(x,y) \\ +2m[\gamma\cdot\partial, \gamma^\mu]D^{\mathrm{F}}_{m,2}(x-y) \\ +2 : \partial_\nu A(x)\gamma^\nu\gamma^\mu S^{\mathrm{F}}_m(x-y)A(y)\psi(y) : \\ -2\mathrm{i} : \partial_\nu A(x)\gamma^\nu\gamma^\mu S^{\mathrm{F}}_m(x-y)B(y)\gamma_5\psi(y) : \\ +2\mathrm{i} : \partial_\nu B(x)\gamma_5\gamma^\nu\gamma^\mu S^{\mathrm{F}}_m(x-y)A(y)\psi(y) : \\ +2 : \partial_\nu B(x)\gamma_5\gamma^\nu\gamma^\mu S^{\mathrm{F}}_m(x-y)A(y)\gamma_5\psi(y) : \\ +2\mathrm{i}m : A(x)\gamma^\mu S^{\mathrm{F}}_m(x-y)A(y)\psi(y) : \\ +2m : A(x)\gamma^\mu S^{\mathrm{F}}_m(x-y)B(y)\gamma_5\psi(y) : \\ -2m : B(x)\gamma_5\gamma^\mu S^{\mathrm{F}}_m(x-y)A(y)\psi(y) : \\ +2\mathrm{i}m : B(x)\gamma_5\gamma^\mu S^{\mathrm{F}}_m(x-y)B(y)\gamma_5\psi(y) : \qquad (4.5.4)$$

In that case one can compute the divergence with respect to the variable x. The result is

$$\frac{\partial}{\partial x^\mu}T(J^\mu(x), T(y)) = -\mathrm{i}\delta(x-y)A(x,y) \\ +2\delta(x-y)\Big[: \partial_\nu A(x)\gamma^\nu A(y)\psi(y) : \\ -\mathrm{i} : \partial_\nu A(x)\gamma^\nu B(y)\gamma_5\psi(y) : \\ +\mathrm{i} : \partial_\nu B(x)\gamma_5\gamma^\nu A(y)\psi(y) : \\ + : \partial_\nu B(x)\gamma_5\gamma^\nu A(y)\gamma_5\psi(y) : \\ +\mathrm{i}m : A(x)A(y)\psi(y) : +2m : A(x)\gamma_5\psi(y) : \\ -m : B(x)\gamma_5 A(y)\psi(y) : \\ +\mathrm{i}m : B(x)\gamma_5 B(y)\gamma_5\psi(y) : \Big] \\ = \mathrm{i}\delta(x-y)\frac{\partial}{\partial y^\mu}T^\mu(y). \qquad (4.5.5)$$

If we make the finite renormalization

$$T(J^\mu(x), T(y)) \longrightarrow T(J^\mu(x), T(y)) + \mathrm{i}\delta(x-y)T^\mu(x), \qquad (4.5.6)$$

then we obtain from the preceding relation

$$\frac{\partial}{\partial x^\mu}T(J^\mu(x), T(y)) + \frac{\partial}{\partial y^\mu}T(T(x), J^\mu(y)) = 0. \qquad (4.5.7)$$

In [14] such a relation is postulated in the general case, i.e.,

$$\sum_{l=1}^n \frac{\partial}{\partial x^\mu_l}T(T(x_1), \ldots, J^\mu(x_l)\ldots, T(x_n)) = 0. \qquad (4.5.8)$$

5 Conclusions

We have proved that the essence of the improved renormalizability properties of supersymmetric models is due to the fact that (4.1.24) is of a purely algebraic nature and so the possible anomalies can be eliminated by a redefinition of the causal splitting. We comment on the corresponding Ward identities following from (4.1.1). If one considers chronological products of Wick submonomials with a proper normalization then one can translate (4.1.1) into equations for the renormalized Feynman amplitudes. One obtains that all expressions from Sect. 4.3 with $d^{\cdots}_{\cdots} \rightarrow t^{\cdots}_{\cdots}$ are null. These are exactly the Ward identities of the Wess–Zumino model [15]. Such type of identities have been extensively studied in [4]. In particular, they impose the behaviour of the Feynman amplitudes described before Theorem 3.3.

A very interesting subject for further investigations is to determine how general is the phenomenon exhibited in this paper, that is the purely algebraic Ward identities.

A reformulation of the preceding analysis in terms of superfields [18,21,20] would also be interesting.

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